2024 年 6 月 21 日　講演者 : 可知 靖之 (Yasuyuki Kachi)

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題 : Stirling's formula, Akiyama-Tanigawa polynomials, and the Riemann–Hurwitz zeta function

講演要旨 : $n! = 1\cdot 2\cdot \cdots \cdot n$ ($n=1,2,3,\cdots$), and $1^k + 2^k + \cdots + n^k$ ($n=1,2,3,\cdots$), are two archetypal examples of integer sequences that had historically enticed mathematicians. The more one looks at them, the more the unfathomable side of the *leitmotif* seems to prevail.

1. When n grows like $10^2, 10^3, 10^4, \cdots$, getting a hold of the exact value of $n!$ becomes far-fetched. It quickly gives way to asymptotic analysis. **Stirling's formula** (1735) asserts

$$(*)\qquad \log(n!) = \log(\sqrt{(2\pi)}) - n + (n+(1/2))\log n + 1/(12n) + O(1/n^3) \quad (n\to\infty).$$

Where does π come from? Can we flirt with the O-term and write it as (a concrete function on n) $+\ O(1/n^r)$ with $r=4,5,6,\cdots$?

2. We all know the fact that $1^k+2^k+\cdots+(n-1)^k$ is a polynomial $f(n)$, provided $k\in\mathbf{Z}$; $k\geq 1$. Interestingly enough, the derivative of that polynomial satisfies $f'(n)-f'(0)=k\cdot(1^{k-1}+\cdots+(n-1)^{k-1})$, an inkling of some *inductive* structure. That structure is *personified* in the Fourier series

$$(**)\qquad f'(x) = -2\cdot(k!)\,\Sigma_{j=1}^{\infty}\sin(2j\pi x + (\pi(1-k)/2))\,/\,(2j\pi)^k \quad (x\in[0,1]).$$

For instance, $1^{12}+\cdots+(n-1)^{12}=f(n)$ is given by

$$f(x) = (1/13)x^{13} - (1/2)x^{12} + \cdots + (5/3)x^3 - (691/2730)x,$$

$$f'(0) = -691/2730 \quad \text{(the } 12^{\text{th}} \textbf{ Bernoulli number}\text{), and}$$

$$f'(x) = -2\cdot(12!)\,\Sigma_{j=1}^{\infty}\cos(2j\pi x)\,/\,(2j\pi)^{12}.$$

For $x=0$, this last equation yields $\zeta(12) := \Sigma_{j=1}^{\infty} j^{-12} = 691\pi^{12}/638512875$.

From the modern perspective, *the Hurwitz zeta function* $\zeta(s;x)$, and its specialization, *the Riemann zeta function* $\zeta(s):=\zeta(s;1)$, encapsulate all of the above. The revved-up versions of $(*)$ and $(**)$ are

$$(\#)\qquad \log(n!) = \log(\sqrt{(2\pi)}) - (n+(1/2)) + (\partial^2/\partial s\,\partial x)\,\zeta(s;x+n)\,\Big|_{s=-1;\,x=1}, \text{ and}$$

$$(\#\#)\qquad \zeta(s;x) = 2\Gamma(1-s)\,\Sigma_{j=1}^{\infty}\sin(2j\pi x+(\pi s/2))\,/\,(2j\pi)^{1-s}, \text{ and}$$

$$(\#\#)'\qquad \zeta(s) = 2\Gamma(1-s)\sin(\pi s/2)\,(2j\pi)^{s-1}\,\zeta(1-s) \quad \textbf{(the Riemann's functional equation)}.$$

In this talk, I walk you through the holy grail of $\zeta(s;x)$, $\zeta(s)$, and the Bernoulli numbers, a rational number sequence inseparably linked to them, by way of mustering a generalization of the celebrated Akiyama–Tanigawa diagram (1997) and a concrete analytic continuation formula of $\zeta(s;x)$ and $\zeta(s)$ which I recently worked out.